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Non-unitary transformation of conservative to dissipative evolutions

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Abstract We briefly review in a general context the theory of non-unitary transformations between conservative dynamics with internal time and dissipative evolutions, developed previously in the context of statistical mechanics and apply the results to show the existence of a similarity transformation between the wave equation and the dissipative telegraphist equation

1. Introduction

The evolution of a conservative system is expressed in terms of a unitary group U_t on a separable Hilbert space $\mathscr{H}$. The time parameter t takes all integer values or all real values. The group property reflects the reversible character of the evolution. Typical examples of conservative systems are the evolution of phase densities of Hamiltonian systems and the evolution of wavefunctions or density operators of quantum systems The hyperbolic wave equation is an infinite-dimensional Hamiltonian system which defines a unitary evolution in the Hilbert space of initial data with finite energy

The conservative systems for which non-unitary transformations to dissipative systems have been constructed are qualified by the existence of internal time.

An internal time operator for the unitary evolution U_t is a self-adjoint operator T with the following property:

$$U_{-t}TU_t = T + tI. \tag{1}$$

The internal time operator is canonically conjugate to the anti-self-adjoint generator L of the unitary group $U_t = e^{Lt}$

$$[L, T] = -I \tag{2}$$

The time operator T allows the attribution of the average age $\langle\psi, T\psi\rangle$ to the states $\psi \in \mathscr{H}$. The average age of the evolved state U_t keeps step with the external clock time t

$$\langle U_t\psi, TU_t\psi\rangle = \langle\psi, T\psi\rangle + t. \tag{3}$$

Internal time operators for unitary dynamics were introduced [1] in the context of unstable dynamical systems of the Kolmogorov type. The unitary group U_t defines the evolution of densities in the Liouville space and the spectral projections of the time operator are the conditional expectations over the time-evolved K-partition. Similarly, quantum systems admit time operators on the Liouville space of density operators if the Hamiltonian has an absolutely continuous spectrum [2] Time operators were also introduced for relativistic fields [3-6].

Conservative systems with internal time are related through non-unitary transformations to a class of autonomous dissipative systems whose evolution is described by a contraction semigroup W_t on a Hilbert space $\mathscr{H}$ that approaches the unique equilibrium monotonically for long times. The time parameter t takes positive integer values or positive real values if evolution is directed towatds the future. The approach to equilibrium is described by the condition

$$\| W_t\psi \|^2 \tag{4}$$

decreases monotonically to zero as $t \to +\infty$. Here the vectors $\psi \in \mathscr{H}$ represent the non-equilibrium deviations of the physical system in consideration. The qualifying feature of such systems is irreversible undirectness of the evolution Typical examples of such dissipative systems include the heat equation, the Boltzmann equation, the stationary Markov processes and the hyperbolic telegraphist equation

The generators of dissipative evolutions $W_t = e^{\Phi t}$ satisfy the property

$$\langle \Phi\psi, \psi \rangle + \langle \psi, \Phi\psi \rangle \leqslant 0 \tag{5}$$

for every ψ in the domain of Φ. This condition is equivalent to the condition that the dissipation part of Φ is negative:

$$\tfrac{1}{2}(\Phi + \Phi^+) \equiv K \leqslant 0 \tag{6}$$

Operators satisfying (5) or (6) were called dissipative operators by Phillips [7] who introduced the concept in his study of dissipative hyperbolic systems such as the Telegraphist equation.

The non-unitary transformation theory of dynamical systems with internal time has been developed as an exact theory of irreversibility in the context of statistical mechanics by Misra, Prigogine and Courbage [8-13] where the positivity of the states ψ is necessary. Here the condition of positivity is relaxed and the non-unitary transformation theory is applied in the more general context of unstable dynamical systems with internal time such as the wave propagation. In this particular case, the wave equation is transformed into the telegraphist equation.

2. The non-unitary transformation

Unitary dynamics U_t with internal time T are related to dissipative evolutions W_t through the intertwining transformation Λ:

$$\Lambda U_t = W_t \Lambda \qquad t \geqslant 0. \tag{7}$$

A non-unitary similarity transformation Λ may be constructed as an operator function of the internal time T:

$$\Lambda = \Lambda(T) = \int_{-\infty}^{+\infty} \Lambda(\tau)\, d\mathbb{P}_\tau. \tag{8}$$

$\mathbb{P}_\tau$ are the spectral projections of the time operator:

$$T = \int_{-\infty}^{+\infty} \tau\, d\mathbb{P}_\tau. \tag{9}$$

The function $\Lambda(\tau)$ can be any positive real function decreasing monotonically to zero as $\tau \to +\infty$. The resulting dissipative evolution

$$W_t = \Lambda U_t \Lambda^{-1} \qquad t \geq 0 \tag{10}$$

arises as a change of representation of the dynamical evolution U_t through the non-unitary transformation Λ. The function $\Lambda(\tau)$ acts as a weight over the contributions of the spectral projections of the time operator to the evolution.

The dissipative character of the evolution W_t follows immediately. For any state ψ in the domain D_Λ of Λ

$$D_\Lambda = \left\{ \psi \in \mathscr{H} \,\middle|\, \int_{-\infty}^{+\infty} |\Lambda(\tau)|^2 \, d\|\mathbb{P}_\tau \psi\|^2 < +\infty \right\}$$

$\tilde{\psi} \equiv \Lambda\psi$ denotes the Λ-transformed state and we have

$$\begin{aligned}
\|W_t\tilde{\psi}\|^2 &= \|\Lambda U_t \Lambda^{-1}\tilde{\psi}\|^2 \\
&= \|U_t U_t^\dagger \Lambda U_t \Lambda^{-1}\tilde{\psi}\|^2 \\
&= \|\Lambda(T+t)\Lambda^{-1}(T)\tilde{\psi}\|^2 \\
&= \int_{-\infty}^{+\infty} \mathrm{d}\tau \left| \frac{\Lambda(\tau+t)}{\Lambda(\tau)} \right|^2 \mathrm{d}\|\mathbb{P}_\tau \tilde{\psi}\|^2
\end{aligned}$$

which decreases monotonically to zero as $t \to +\infty$.

If the function $\Lambda(\tau)$ is bounded then the operator $\Lambda(T)$ is a bounded self-adjoint operator. However, the inverse operator $\Lambda^{-1}(T)$ cannot be bounded.

If the function $\Lambda(\tau)$ is not bounded, then the operator $\Lambda(T)$ is self-adjoint if the domain D_Λ is dense [14].

If the domain D_Λ is not dense then the contraction semigroup is defined on the subspace of vectors $\tilde{\psi}$ for which $U_t\Lambda^{-1}\tilde{\psi}$ is in the domain D_Λ

Typical examples of the weighting function $\Lambda(\tau)$ are the logarithmically concave functions of the form

$$\Lambda(\tau) = \mathrm{e}^{-\gamma(\tau)}$$

where $\gamma(\tau)$ is a convex function.

For the simple choice

$$\Lambda(\tau) = \mathrm{e}^{-\gamma\tau} \tag{11}$$

where γ is a constant parameter the unbounded operator $\Lambda(T) = \mathrm{e}^{-\gamma T}$ is densely defined.

To see this let us first note that for any vector ψ, the measures $\mathrm{d}\|\mathbb{P}_\tau\psi\|^2$ associated with the spectral projections $\mathbb{P}_\tau$ of T are absolutely continuous because the time operator has absolutely continuous spectrum. Therefore they can be written as

$$\mathrm{d}\|\mathbb{P}_\tau\psi\|^2 = \rho_\psi(\tau)\,\mathrm{d}\tau$$

where $\rho_\psi(\tau)$ is the corresponding spectral density defined as the Radon–Nicodym derivative of $\mathrm{d}\|\mathbb{P}_\tau\psi\|^2$ with respect to the Lebesque measure $\mathrm{d}\tau$.

The set D of all vectors $\psi \in \mathscr{H}$ for which the corresponding spectral density $\rho_\psi(\tau)$ belongs to the space spanned by all functions of the form $\mathrm{e}^{-\tau^2}f(\tau)$, $f(\tau)$ being a polynomial in the real variable τ, is dense.

It can be easily seen that the domain D is included in the domains of both Λ and Λ^{-1} and that D is also invariant under U_t. For example,

$$\|\Lambda\psi\|^2 = \int_{-\infty}^{+\infty} e^{-2\gamma\tau}\rho_\psi(\tau)\,d\tau$$

which is finite for $\rho_\psi(\tau)$ of the form $e^{-\tau^2}f(\tau)$ where $f(\tau)$ is a polynomial on the real variable τ.

Therefore, the operators Λ and Λ^{-1} as well as the semigroup $W_t = \Lambda U_t \Lambda^{-1}$ are defined on the dense domain D and since W_t is a contraction semigroup, its domain can be extended to the entire Hilbert space $\mathscr{H}$.

The generator Φ of the Λ-transformed evolution

$$W_t = \Lambda U_t \Lambda^{-1} = e^{\Phi t}$$

is the dissipative operator

$$\Phi = \Lambda L \Lambda^{-1} = L + K \tag{12}$$

with

$$K = K(T) = \Lambda'(T)\Lambda^{-1}(T) = \int_{-\infty}^{+\infty} \frac{\Lambda'(\tau)}{\Lambda(\tau)}\,d\mathbb{P}_\tau$$

the negative dissipation part (6) of Φ.

The proof follows immediately. Consider any vector ψ in the domain D_Φ of the generator Φ The strong limit [15] definition of the generator means:

$$\lim_{t\to 0_+} \left\| \frac{\Lambda U_t \Lambda^{-1} - I}{t}\psi - \Phi\psi \right\| = 0$$

$$\Leftrightarrow \lim_{t\to 0_+} \left\| \frac{U_t(U_t^{\dagger}\Lambda U_t)\Lambda^{-1} - I}{t}\psi - \Phi\psi \right\| = 0$$

$$\Leftrightarrow \lim_{t\to 0_+} \left\| \frac{U_t(\Lambda(T+t))\Lambda^{-1} - I}{t}\psi - \Phi\psi \right\| = 0$$

$$\Leftrightarrow \lim_{t\to 0_+} \left\| \frac{U_t(\Lambda(T+t) - \Lambda(T))\Lambda^{-1}}{t}\psi + \frac{U_t - I}{t}\psi - \Phi\psi \right\| = 0.$$

Since $[U_t(\Lambda(T+t)-\Lambda(T))\Lambda^{-1}]/t$ converges strongly to $\Lambda'(T)\Lambda^{-1}$ and $(U_t - I)/t$ converges strongly to the anti-self-adjoint generator L, the generator Φ is the sum $L + \Lambda'(T)\Lambda^{-1}$ in the common domain of L and $\Lambda'(T)\Lambda^{-1}$:

$$D_\Phi = D_L \cap \left\{ \psi \,\middle|\, \int \left|\frac{\Lambda'(\tau)}{\Lambda(\tau)}\right|^2 d\|\mathbb{P}_\tau\psi\|^2 < +\infty \right\}.$$

The domain D_Φ of Φ depends in general upon the form of Λ. One needs to prove therefore that the domain D_Φ is dense for any concrete choice of the Λ-transformation. For example, if $\Lambda'(\tau)/\Lambda(\tau)$ is a bounded real function, then $D_\Phi = D_L$, which is dense, as L is an anti-self-adjoint operator. This is indeed the case for the special choice (11) of the Λ-transformation. For this choice the dissipative generator Φ is

$$\Phi = \Lambda L \Lambda^{-1} = L - \gamma I. \tag{13}$$

In the case of Kolmogorov systems, where the spectral projections of the time operator are the conditional expectations over the time-evolved K-partition, the non-unitary similarity Λ leads to a Markov process. This was first proved for the Baker shift [8] and then for Bernoulli [9] and Kolmogorov [10] systems.

An intertwining transformation is also provided by the spectral projection $\mathbb{P}_0$ or any of the spectral projections $\mathbb{P}_\tau$ of the time operator (9). The proof is given [11] for K-systems where the condition of positivity preserving for the projections $\mathbb{P}_\tau$ is also assumed. The resulting dissipative evolution

$$W_t = \mathbb{P} U_t \mathbb{P}$$

where $\mathbb{P}$ is any of the projections $\mathbb{P}_\tau$, is the restriction of the dynamics U_t into the $\mathbb{P}$-subspace.

It is also of interest to mention what is known on the inverse problem, namely, what kind of unitary dynamics can be transformed into dissipative evolutions through intertwining transformations:

(i) If the intertwining transformation is a projection, the condition that the unitary dynamics admits a time operator is necessary. In this case the unitary dynamics is a dilation [16] of the dissipative evolution. The corresponding theorem was stated in [11] and proved in [13] for Kolmogorov systems, where the important question of positivity of states arises.

(ii) If the intertwining transformation Λ is a non-unitary similarity, then the operator $M = \Lambda^+\Lambda$ is a Lyapounov operator [1] for the unitary dynamics and it is only known [1] that the generator of the unitary evolution must have absolutely continuous spectrum. This condition is of course weaker than the existence of the time operator. In the case of unitary evolutions of phase densities, this condition implies the strong mixing property, which is weaker than the Kolmogorov property.

3. Non-unitary transformation of the wave equation

The wave equation

$$\partial_t^2 \psi = \Delta \psi \tag{14}$$

is a conservative dynamical system with internal time [3–6] The particular choice $\Lambda(\tau) = e^{-\gamma\tau}$ for the weight function of the non-unitary similarity transformation transforms the wave equation (14) into the dissipative telegraphist equation:

$$\partial_t^2 \tilde{\psi} = \Delta \tilde{\psi} - 2\gamma \partial_t \tilde{\psi} - \gamma^2 \tilde{\psi} \qquad \tilde{\psi} \equiv \Lambda \psi. \tag{15}$$

The wave equation is an infinite-dimensional Hamiltonian system:

$$\partial_t \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}. \tag{16}$$

The space of square integrable initial data $(\psi, \dot{\psi})$ with finite energy

$$H = \int d(x) \tfrac{1}{2} (\dot{\psi}^2(x) + |\nabla \psi(x)|^2) \tag{17}$$

is the Sobolev Hilbert space $\mathscr{H}$ with the scalar product

$$\left\langle \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix}, \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} \right\rangle = \int d(x) \tfrac{1}{2} (\dot{\psi}^* \dot{\phi} + \nabla \psi^* \cdot \nabla \phi). \tag{18}$$

The time evolution U_t of the Cauchy data in $\mathscr{H}$ [17] is

$$U_t \begin{pmatrix} \psi \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} \cos Bt & B^{-1} \sin Bt \\ -B \sin Bt & \cos Bt \end{pmatrix} \tag{19}$$

where $B=\sqrt{-\Delta}$. The group U_t preserves the energy (17) and the scalar product (18) The anti-self-adjoint generator of the unitary group $U_t = e^{Lt}$ is

$$L=\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}. \tag{20}$$

An internal time operator for the wave equation has been constructed [3-6] from the representations of the Poincaré algebra. The explicit form of the time operator is

$$T\begin{pmatrix} \psi(x) \\ \dot\psi(x) \end{pmatrix} = -\begin{pmatrix} \sum_{a=1}^{3} x^a \frac{\partial}{\partial x^a} \Delta^{-1}\dot\psi(x) \\ \psi(x) + \sum_{a=1}^{3} x^a \frac{\partial}{\partial x^a}\psi(x) \end{pmatrix}. \tag{21}$$

For the particular choice $\Lambda(\tau)=e^{-\gamma\tau}$ for the weight function, the generator L given by (20) is transformed into the generator Φ through (13).

The Λ-transformed wave equation is the telegraphist equation:

$$\partial_t\begin{pmatrix} \tilde\psi \\ \tilde{\dot\psi} \end{pmatrix} = \left[\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} - \gamma\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right]\begin{pmatrix} \tilde\psi \\ \tilde{\dot\psi} \end{pmatrix}$$

$$\Leftrightarrow \partial_t\begin{pmatrix} \tilde\psi \\ \tilde{\dot\psi} \end{pmatrix} = \begin{pmatrix} -\gamma\tilde\psi + \tilde{\dot\psi} \\ \Delta\tilde\psi - \gamma\tilde{\dot\psi} \end{pmatrix}$$

$$\Leftrightarrow \partial_t^2\tilde\psi = \Delta\tilde\psi - 2\gamma\partial_t\tilde\psi - \gamma^2\tilde\psi.$$

As a result of the particular construction of the Λ-transformation, the coefficients -2γ and $-\gamma^2$ of the last two terms of (15) are not independent. The derived equation (15) is not therefore the most general telegraphist equation.

The telegraphist equation is a dissipative hyperbolic equation describing waves propagating with exponential dissipation of the amplitude. This equation arises also as a relativistic heat equation if we introduce finite propagation velocity into the heat equation, which is Galilei invariant and assumes infinite propagation velocity [18].

4. Concluding remarks

The non-unitary transformation of dynamical systems with internal time arises as a limitation on the contributions of the future spectral projections of the time operator to the dynamical evolution. In the case of projection $\mathbb{P}_{\tau_1}$ all future projections $\mathbb{P}_\tau$, $\tau > \tau_1$ are not realized and the evolution is restricted to the $\mathbb{P}_{\tau_1}$-subspace. In the case of similarity $\Lambda(T)$, the future projections are weighted with the decreasing function $\Lambda(\tau)$.

This interpretation of Λ-transformation has a clear meaning for Kolmogorov systems where the spectral projections of the time operator are coarse-graining projections of conditional expectation over the cells of the K-partition. The progressive refinement of the K-partition expresses the dynamical instability.

The resulting dissipative Markov semigroup W_t is a best prediction for U_t, made by an observer who cannot follow precisely the progressive refinements of the K-partition *ad infinitum*.

In fact, this construction has been proposed as an exact theory of irreversibility [7-12] in the context of statistical mechanics where the positivity of the states is important.

In this work the condition of positivity is dropped This permits us to consider more general functions $\Lambda(\tau)$ which bring about the similarity transformation between conservative and dissipative evolutions.

Λ-transformations may also be seen as 'transmutations' [19] relating properties to conservative equations with properties of dissipative equations.

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